Extra Practice Problems 2

Here are some extra practice problems on topics that were popular on the Google Moderator site. We'll release solutions to these problems on Monday.

Problem One: Binary Relations

i. Let Σ be some alphabet and L be a language over Σ. Two strings $x, y \in \Sigma^*$ are called *indistinguishable* relative to L, denoted $x \equiv_L y$, iff for every $w \in \Sigma^*$ we have $xw \in L$ iff $yw \in L$. This relation arises in a more complete version of the Myhill-Nerode theorem. Prove that $\equiv_L$ is an equivalence relation over Σ^*.

ii. Prove that a binary relation is a total order iff it is total, antisymmetric, and transitive.

iii. How many equivalence relations are there over the set $\{a, b, c\}$?

Problem Two: Injections, Surjections, and Bijections

i. Find functions $f : A \to B$ and $g : B \to C$ where $g \circ f$ is a bijection but neither f nor g are bijections.

ii. An *involution* is a function $f : A \to A$ where $f(f(x)) = x$. Prove that all involutions are bijections.

Problem Three: Regular and Nonregular Languages

i. Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ and let $L = \{ \mathtt{a}^n\mathtt{b}^m \mid n, m \in \mathbb{N} \text{ and } n \neq m \}$. Prove that L is not regular.

ii. Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ and let $L = \{ w \in \Sigma^* \mid w \text{ is a palindrome} \}$. Prove that L is not regular.

iii. Let $\Sigma = \{\mathtt{a}\}$ and let $L = \{ w \in \Sigma^* \mid w \text{ is a palindrome} \}$. Prove that L *is* regular.

Problem Four: Closure Properties and Nonregular Languages

The regular languages are closed under intersection; if L_1 and L_2 are regular, then $L_1 \cap L_2$ is regular as well.

i. Is the converse of this statement true? That is, if $L_1 \cap L_2$ is regular, then are L_1 and L_2 regular? Prove or disprove this statement.

The fact that regular languages are closed under intersection can be used to prove that various languages are not regular without using the Myhill-Nerode theorem. For example, let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ and let $L = \{ w \in \Sigma^* \mid w \text{ has the same number of } \mathtt{a}\text{'s and } \mathtt{b}\text{'s} \}$. This language is similar to the language $L' = \{ \mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N} \}$, which we know isn't regular. Using this fact, it's possible to prove that L can't be regular.

ii. Find a regular language R such that $L \cap R = L'$.

iii. Using your result from (ii), prove that L is not regular.

Problem Five: R, RE, co-RE Languages

i. Prove that the **R** languages are closed under intersection.

ii. Prove that the **RE** languages are closed under intersection.

iii. Prove that the co-**RE** languages are closed under intersection.

Problem Six: R, RE, and co-RE Languages II

In lecture, we sketched a proof that if M is a recognizer for L, then the machine M' formed by swapping the accept and reject states of M is a co-recognizer for $\overline{L}$. However, the machine M' won't in general be a recognizer for $\overline{L}$.

Prove that the machine M' formed by swapping the accept and reject states of M is a recognizer for the language $\overline{L}$ iff M is a decider.

Problem Seven: Mapping Reducibility

i. A *nontrivial language* is a language other than Ø and Σ^*. Prove that all nontrivial decidable languages are mapping reducible to one another.

ii. Find an example of an **RE** language and a co-**RE** language that are mapping reducible to one another.

iii. Prove that $L \leq_M \Sigma^*$ iff $L = \Sigma^*$.